Normal Forms and Satisfiability

COM 270: Introduction to Automated Deduction

Sasha Fedchin[1]

[1]Sotware Engineering Department
AUCA

Normal Forms: Motivation

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- Assuming that the input is in the normal form facilitates automation.
- The two key normal norms for propositional logic are:
  - Conjunctive Normal Form
  - Disjunctive Normal Form

Conjunctive Normal Form

- A formula is in conjunctive normal form (CNF) if it is a conjunction of clauses, where each clause is a disjunction of literals. Examples:
  - $(p \vee q) \wedge (q \vee \neg r)$
  - $(p \vee \neg p) \wedge (a \vee d \vee \neg a) \wedge c$
  - $p \wedge q$
  - $p \vee \neg p \equiv \mathbf{T}$

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- A formula is a tautology if and only if every clause in its CNF representation contains a pair of complementary literals.
- Any clause in the CNF that does not have such a pair gives a variable truth assignment (a.k.a. interpretation) that falsifies the formula.

Disjunctive Normal Form

- A formula is in disjunctive normal form (DNF) if it is a disjunction of clauses, where each clause is a conjunction of literals.
  - $(p \wedge q) \vee (q \wedge \neg r)$
  - $p \vee q$
  - $(p \wedge \neg p) \vee (a \wedge d \wedge \neg a) \vee c$
  - $p \wedge \neg p \equiv \mathbf{F}$

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- A formula is false under every interpretation if and only if every clause in its DNF representation contains a pair of complementary literals.
- Any clause in the DNF that does not have such a pair gives an interpretation that satisfies the formula.

Duality of Satisfiability and Tautology

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- For example, $\{p \mapsto \mathbf{T}, q \mapsto \mathbf{F}\} \models p \wedge \neg q$.
- A formula F is satisfiable if and only if $I \models F$ for some I.
- F is a tautology if $\neg F$ is unsatisfiable.
- A SAT solver can find a satisfying interpretation or it can prove that a formula is a tautology.

Recall Some Useful Logical Equivalences

$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$p \equiv \neg(\neg p)$	Double negation law
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws

$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

$p \rightarrow q \equiv \neg p \vee q$	Implication elimination law
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$ $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$	Bicondition elimination laws

Converting to CNF (DNF)

Any propositional formula can be converted to CNF (DNF) using the following algorithm:

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- Remove all double negations (the result is sometimes called negation normal form).

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Any propositional formula can be converted to CNF (DNF) using the following algorithm:

- Eliminate all implications and biconditionals.
- Apply De Morgan's laws to move negation inward.
- Remove all double negations (the result is sometimes called negation normal form).
- Distribute over conjunction (disjunction).

Conversion to CNF

$$((p \to q) \wedge \neg q) \to \neg p \equiv \neg((p \to q) \wedge \neg q) \vee \neg p \qquad \text{by eliminating implication}$$

Conversion to CNF

$$
\begin{aligned}
((p \to q) \land \neg q) \to \neg p &\equiv \neg((p \to q) \land \neg q) \lor \neg p && \text{by eliminating implication} \\
&\equiv \neg((\neg p \lor q) \land \neg q) \lor \neg p && \text{by eliminating implication}
\end{aligned}
$$

Conversion to CNF

$$\begin{aligned}
((p \to q) \wedge \neg q) \to \neg p &\equiv \neg((p \to q) \wedge \neg q) \vee \neg p && \text{by eliminating implication} \\
&\equiv \neg((\neg p \vee q) \wedge \neg q) \vee \neg p && \text{by eliminating implication} \\
&\equiv \neg(\neg p \vee q) \vee \neg\neg q \vee \neg p && \text{by the first De Morgan law}
\end{aligned}$$

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&\equiv \neg(\neg p \vee q) \vee \neg\neg q \vee \neg p && \text{by the first De Morgan law} \\
&\equiv (\neg\neg p \wedge \neg q) \vee \neg\neg q \vee \neg p && \text{by the second De Morgan law}
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&\equiv (\neg\neg p \wedge \neg q) \vee \neg\neg q \vee \neg p && \text{by the second De Morgan law} \\
&\equiv (p \wedge \neg q) \vee \neg\neg q \vee \neg p && \text{by the double negation law}
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Conversion to CNF

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\begin{aligned}
((p \to q) \land \neg q) \to \neg p &\equiv \neg((p \to q) \land \neg q) \lor \neg p && \text{by eliminating implication} \\
&\equiv \neg((\neg p \lor q) \land \neg q) \lor \neg p && \text{by eliminating implication} \\
&\equiv \neg(\neg p \lor q) \lor \neg\neg q \lor \neg p && \text{by the first De Morgan law} \\
&\equiv (\neg\neg p \land \neg q) \lor \neg\neg q \lor \neg p && \text{by the second De Morgan law} \\
&\equiv (p \land \neg q) \lor \neg\neg q \lor \neg p && \text{by the double negation law} \\
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&\equiv (p \wedge \neg q) \vee \neg\neg q \vee \neg p && \text{by the double negation law} \\
&\equiv (p \wedge \neg q) \vee q \vee \neg p && \text{by the double negation law} \\
&\equiv q \vee (p \wedge \neg q) \vee \neg p && \text{by the first commutative law}
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&\equiv (p \land \neg q) \lor q \lor \neg p && \text{by the double negation law} \\
&\equiv q \lor (p \land \neg q) \lor \neg p && \text{by the first commutative law} \\
&\equiv ((p \lor q) \land (\neg q \lor q)) \lor \neg p && \text{by the first distributive law}
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&\equiv \mathbf{T}
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CNF (DNF) Conversion can be infeasible inpractice

$$(a \wedge b) \vee (c \wedge d) \vee (e \wedge f) \vee (g \wedge h)$$

$$\equiv (a \vee (c \wedge d) \vee (e \wedge f) \vee (g \wedge h)) \wedge (b \vee (c \wedge d) \vee (e \wedge f) \vee (g \wedge h))$$

$$\equiv (a \vee c \vee (e \wedge f) \vee (g \wedge h)) \wedge (a \vee d \vee (e \wedge f) \vee (g \wedge h))$$
$$\wedge (b \vee c \vee (e \wedge f) \vee (g \wedge h)) \wedge (b \vee d \vee (e \wedge f) \vee (g \wedge h))$$

$$\equiv (a \vee c \vee e \vee (g \wedge h)) \wedge (a \vee c \vee f \vee (g \wedge h)) \wedge (a \vee d \vee e \vee (g \wedge h)) \wedge (a \vee d \vee f \vee (g \wedge h))$$
$$\wedge (b \vee c \vee e \vee (g \wedge h)) \wedge (b \vee c \vee f \vee (g \wedge h)) \wedge (b \vee d \vee e \vee (g \wedge h)) \wedge (b \vee d \vee f \vee (g \wedge h)$$

$$\equiv (a \vee c \vee e \vee g) \wedge (a \vee c \vee e \vee h) \wedge (a \vee c \vee f \vee g) \wedge (a \vee c \vee f \vee h)$$
$$\wedge (a \vee d \vee e \vee g) \wedge (a \vee d \vee e \vee h) \wedge (a \vee d \vee f \vee g) \wedge (a \vee d \vee f \vee h)$$
$$\wedge (b \vee c \vee e \vee g) \wedge (b \vee c \vee e \vee h) \wedge (b \vee c \vee f \vee g) \wedge (b \vee c \vee f \vee h)$$
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$\equiv (a \vee (c \wedge d) \vee (e \wedge f) \vee (g \wedge h)) \wedge (b \vee (c \wedge d) \vee (e \wedge f) \vee (g \wedge h))$

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$\wedge (b \vee c \vee (e \wedge f) \vee (g \wedge h)) \wedge (b \vee d \vee (e \wedge f) \vee (g \wedge h))$

$\equiv (a \vee c \vee e \vee (g \wedge h)) \wedge (a \vee c \vee f \vee (g \wedge h)) \wedge (a \vee d \vee e \vee (g \wedge h)) \wedge (a \vee d \vee f \vee (g \wedge h))$
$\wedge (b \vee c \vee e \vee (g \wedge h)) \wedge (b \vee c \vee f \vee (g \wedge h)) \wedge (b \vee d \vee e \vee (g \wedge h)) \wedge (b \vee d \vee f \vee (g \wedge h)$

$\equiv (a \vee c \vee e \vee g) \wedge (a \vee c \vee e \vee h) \wedge (a \vee c \vee f \vee g) \wedge (a \vee c \vee f \vee h)$
$\wedge (a \vee d \vee e \vee g) \wedge (a \vee d \vee e \vee h) \wedge (a \vee d \vee f \vee g) \wedge (a \vee d \vee f \vee h)$
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$\wedge (b \vee d \vee e \vee g) \wedge (b \vee d \vee e \vee h) \wedge (b \vee d \vee f \vee g) \wedge (b \vee d \vee f \vee h)$

Input: 4 Conjunctive clauses. Output: $2^4 = 16$ disjunctive clauses.

Tseytin Transformation

- Original Formula: $(a \wedge b) \vee (c \wedge d) \vee (e \wedge f) \vee (g \wedge h)$

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- Introduce a new boolean variable for each subformula:
  - $w \leftrightarrow (a \wedge b)$
  - $x \leftrightarrow (c \wedge d))$
  - $y \leftrightarrow (e \wedge f))$
  - $z \leftrightarrow (g \wedge h))$

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  - $w \leftrightarrow (a \wedge b) \equiv (w \rightarrow (a \wedge b)) \wedge ((a \wedge b) \rightarrow w))$
  - $x \leftrightarrow (c \wedge d)) \equiv (x \rightarrow (c \wedge d) \wedge ((c \wedge d) \rightarrow x))$
  - $y \leftrightarrow (e \wedge f)) \equiv (y \rightarrow (e \wedge f) \wedge ((e \wedge f) \rightarrow y))$
  - $z \leftrightarrow (g \wedge h)) \equiv (z \rightarrow (g \wedge h) \wedge ((g \wedge h) \rightarrow z))$

Tseytin Transformation

- Original Formula: $(a \land b) \lor (c \land d) \lor (e \land f) \lor (g \land h)$
- Introduce a new boolean variable for each subformula:
  - $w \leftrightarrow (a \land b) \equiv (w \rightarrow (a \land b)) \land ((a \land b) \rightarrow w)) \equiv (\neg w \lor a) \land (\neg w \lor b) \land (\neg a \lor \neg b \lor w)$
  - $x \leftrightarrow (c \land d)) \equiv (x \rightarrow (c \land d) \land ((c \land d) \rightarrow x)) \equiv (\neg x \lor c) \land (\neg x \lor d) \land (\neg c \lor \neg d \lor x)$
  - $y \leftrightarrow (e \land f)) \equiv (y \rightarrow (e \land f) \land ((e \land f) \rightarrow y)) \equiv (\neg y \lor e) \land (\neg y \lor f) \land (\neg e \lor \neg f \lor y)$
  - $z \leftrightarrow (g \land h)) \equiv (z \rightarrow (g \land h) \land ((g \land h) \rightarrow z)) \equiv (\neg z \lor g) \land (\neg z \lor h) \land (\neg g \lor \neg h \lor z)$

Tseytin Transformation

- Original Formula: $(a \wedge b) \vee (c \wedge d) \vee (e \wedge f) \vee (g \wedge h)$
- Introduce a new boolean variable for each subformula:
  - $w \leftrightarrow (a \wedge b) \equiv (w \rightarrow (a \wedge b)) \wedge ((a \wedge b) \rightarrow w)) \equiv (\neg w \vee a) \wedge (\neg w \vee b) \wedge (\neg a \vee \neg b \vee w)$
  - $x \leftrightarrow (c \wedge d)) \equiv (x \rightarrow (c \wedge d) \wedge ((c \wedge d) \rightarrow x)) \equiv (\neg x \vee c) \wedge (\neg x \vee d) \wedge (\neg c \vee \neg d \vee x)$
  - $y \leftrightarrow (e \wedge f)) \equiv (y \rightarrow (e \wedge f) \wedge ((e \wedge f) \rightarrow y)) \equiv (\neg y \vee e) \wedge (\neg y \vee f) \wedge (\neg e \vee \neg f \vee y)$
  - $z \leftrightarrow (g \wedge h)) \equiv (z \rightarrow (g \wedge h) \wedge ((g \wedge h) \rightarrow z)) \equiv (\neg z \vee g) \wedge (\neg z \vee h) \wedge (\neg g \vee \neg h \vee z)$
- Now combine this together into an equisatisfiable formula: $(w \vee x \vee y \vee z) \wedge (\neg w \vee a) \wedge (\neg w \vee b) \wedge (\neg a \vee \neg b \vee w) \wedge (\neg x \vee c) \wedge (\neg x \vee d) \wedge (\neg c \vee \neg d \vee x) \wedge (\neg y \vee e) \wedge (\neg y \vee f) \wedge (\neg e \vee \neg f \vee y) \wedge (\neg z \vee g) \wedge (\neg z \vee h) \wedge (\neg g \vee \neg h \vee z)$

Tseytin Transformation

- Original Formula: $(a \wedge b) \vee (c \wedge d) \vee (e \wedge f) \vee (g \wedge h)$
- Introduce a new boolean variable for each subformula:
  - $w \leftrightarrow (a \wedge b) \equiv (w \rightarrow (a \wedge b)) \wedge ((a \wedge b) \rightarrow w)) \equiv (\neg w \vee a) \wedge (\neg w \vee b) \wedge (\neg a \vee \neg b \vee w)$
  - $x \leftrightarrow (c \wedge d)) \equiv (x \rightarrow (c \wedge d) \wedge ((c \wedge d) \rightarrow x)) \equiv (\neg x \vee c) \wedge (\neg x \vee d) \wedge (\neg c \vee \neg d \vee x)$
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- Now combine this together into an equisatisfiable formula: $(w \vee x \vee y \vee z) \wedge (\neg w \vee a) \wedge (\neg w \vee b) \wedge (\neg a \vee \neg b \vee w) \wedge (\neg x \vee c) \wedge (\neg x \vee d) \wedge (\neg c \vee \neg d \vee x) \wedge (\neg y \vee e) \wedge (\neg y \vee f) \wedge (\neg e \vee \neg f \vee y) \wedge (\neg z \vee g) \wedge (\neg z \vee h) \wedge (\neg g \vee \neg h \vee z)$
- Input: 4 Conjunctive clauses. Output: $1 + 3 * 4 = 13$ disjunctive clauses.

Tseytin Transformation

- Original Formula: $(a \wedge b) \vee (c \wedge d) \vee (e \wedge f) \vee (g \wedge h)$
- Introduce a new boolean variable for each subformula:
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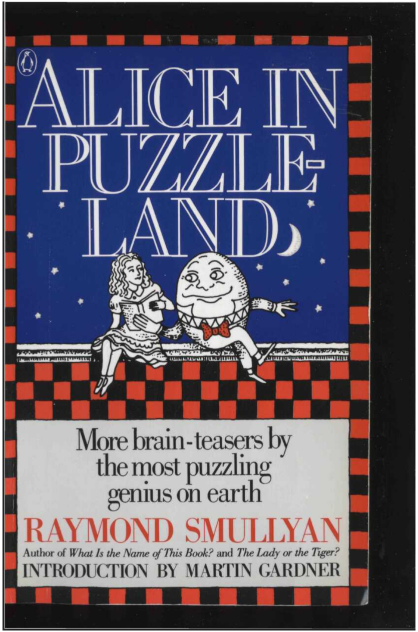
- Input: 4 Conjunctive clauses. Output: $1 + 3 * 4 = 13$ disjunctive clauses.
- Note that the formula is **not** logically equivalent. E.g. Tseytin transformation would map $(\neg a \vee b) \leftrightarrow (a \rightarrow b)$ to $(\neg x \vee \neg a \vee b) \wedge (a \vee x) \wedge (\neg b \vee x) \wedge (\neg y \vee \neg a \vee b) \wedge (a \vee y) \wedge (\neg b \vee y) \wedge (\neg x \vee y) \wedge (\neg y \vee x)$. The former is a tautology, the latter is not.

Digression: Check out this Book!



Encoding Text into Logic

From *Alice in Puzzleland* by Raymond Smullyan:

"The only possible suspects were the Cook, the Duchess, and the Cheshire Cat.
"The Cheshire Cat stole it!" said the Duchess at the trial.
"Oh, yes, I stole it!" said the Cheshire Cat with a grin.
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As it turned out, the thief had lied and at least one of the others had told the truth.
Who stole the cookbook?"

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Let D stand for "The Duchess stole the cookbook".
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Let K stand for "The Cook stole the cookbook".
Let d stand for "The Duchess told the truth".
Let c stand for "The Cheshire Cat told the truth".
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SAT Encoding and Tseytin Transformation Example

Original Formula:

$\land (d \leftrightarrow C)$

$\land (c \leftrightarrow C)$

$\land (k \leftrightarrow \neg K)$

$\land (D \rightarrow (\neg d \land (k \lor c)))$

$\land (K \rightarrow (\neg k \land (c \lor d)))$

$\land (C \rightarrow (\neg c \land (d \lor k)))$

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- Introduce variables x, y, z, X, Y, Z:
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  - $y \leftrightarrow (c \lor d)$ and $Y \leftrightarrow (\neg k \land y)$
  - $z \leftrightarrow (d \lor k)$ and $Z \leftrightarrow (\neg c \land z)$

New Formula (Eliminated Biconditionals):

$\land (d \rightarrow C) \land (C \rightarrow d)$

$\land (c \rightarrow C) \land (C \rightarrow c)$

$\land (k \rightarrow K) \land (K \rightarrow k)$

$\land (D \rightarrow X)$

$\land (K \rightarrow Y)$

$\land (C \rightarrow Z)$

$\land (x \rightarrow (k \lor c)) \land ((k \lor c) \rightarrow x)$

$\land (y \rightarrow (c \lor d)) \land ((c \lor d) \rightarrow y)$

$\land (z \rightarrow (d \lor k)) \land ((d \lor k) \rightarrow z)$

$\land (X \rightarrow (\neg d \land x)) \land ((\neg d \land x) \rightarrow X)$

$\land (Y \rightarrow (\neg k \land y)) \land ((\neg k \land y) \rightarrow Y)$

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  - $z \leftrightarrow (d \vee k)$ and $Z \leftrightarrow (\neg c \wedge z)$

New Formula (Eliminated Implications):

$\wedge (\neg d \vee C) \wedge (\neg C \vee d)$

$\wedge (\neg c \vee C) \wedge (\neg C \vee c)$

$\wedge (\neg k \vee K) \wedge (\neg K \vee k)$

$\wedge (\neg D \vee X)$

$\wedge (\neg K \vee Y)$

$\wedge (\neg C \vee Z)$

$\wedge (\neg x \vee (k \vee c)) \wedge (\neg(k \vee c) \vee x)$

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- Introduce variables x, y, z, X, Y, Z:
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  - $y \leftrightarrow (c \vee d)$ and $Y \leftrightarrow (\neg k \wedge y)$
  - $z \leftrightarrow (d \vee k)$ and $Z \leftrightarrow (\neg c \wedge z)$

New Formula (NNF):

$\wedge (\neg d \vee C) \wedge (\neg C \vee d)$

$\wedge (\neg c \vee C) \wedge (\neg C \vee c)$

$\wedge (\neg k \vee K) \wedge (\neg K \vee k)$

$\wedge (\neg D \vee X)$

$\wedge (\neg K \vee Y)$

$\wedge (\neg C \vee Z)$

$\wedge (\neg x \vee (k \vee c)) \wedge ((\neg k \wedge \neg c) \vee x)$

$\wedge (\neg y \vee (c \vee d)) \wedge ((\neg c \wedge \neg d) \vee y)$

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$\wedge (\neg X \vee (\neg d \wedge x)) \wedge (d \vee \neg x \vee X)$

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$\wedge (K \rightarrow (\neg k \wedge (c \vee d)))$

$\wedge (C \rightarrow (\neg c \wedge (d \vee k))$

$\wedge (D \vee C \vee K)$

- Introduce variables x, y, z, X, Y, Z:
  - $x \leftrightarrow (k \vee c)$ and $X \leftrightarrow (\neg d \wedge x)$
  - $y \leftrightarrow (c \vee d)$ and $Y \leftrightarrow (\neg k \wedge y)$
  - $z \leftrightarrow (d \vee k)$ and $Z \leftrightarrow (\neg c \wedge z)$

New Formula (CNF):

$\wedge (\neg d \vee C) \wedge (\neg C \vee d)$

$\wedge (\neg c \vee C) \wedge (\neg C \vee c)$

$\wedge (\neg k \vee \neg K) \wedge (K \vee k)$

$\wedge (\neg D \vee X)$

$\wedge (\neg K \vee Y)$

$\wedge (\neg C \vee Z)$

$\wedge (\neg x \vee k \vee c) \wedge (\neg k \vee x) \wedge (\neg c \vee x)$

$\wedge (\neg y \vee c \vee d) \wedge (\neg c \vee y) \wedge (\neg d \vee y)$

$\wedge (\neg z \vee d \vee k) \wedge (\neg d \vee z) \wedge (\neg k \vee z)$

$\wedge (\neg X \vee \neg d) \wedge (\neg X \vee x)) \wedge (d \vee \neg x \vee X)$

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  - $y \leftrightarrow (c \vee d)$ and $Y \leftrightarrow (\neg k \wedge y)$
  - $z \leftrightarrow (d \vee k)$ and $Z \leftrightarrow (\neg c \wedge z)$
- Solution (obtained by a SAT solver): $D \wedge \neg K \wedge \neg C \wedge \neg d \wedge \neg c \wedge k \wedge X \wedge x \wedge \neg Y \wedge \neg y \wedge Z \wedge z$

New Formula (CNF):

$\wedge (\neg d \vee C) \wedge (\neg C \vee d)$

$\wedge (\neg c \vee C) \wedge (\neg C \vee c)$

$\wedge (\neg k \vee \neg K) \wedge (K \vee k)$

$\wedge (\neg D \vee X)$

$\wedge (\neg K \vee Y)$

$\wedge (\neg C \vee Z)$

$\wedge (\neg x \vee k \vee c) \wedge (\neg k \vee x) \wedge (\neg c \vee x)$

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$\wedge (\neg Z \vee \neg c) \wedge (\neg Z \vee z)) \wedge (c \vee \neg z \vee Z)$

$\wedge (D \vee C \vee K)$

Smullyan's Solution

- "It is impossible that the Cheshire Cat stole it, because the thief would then be telling the truth. Therefore the Cheshire Cat didn't steal it (and the Cat and the Duchess were both lying). If the Cook stole it, then all three would be lying, which is contrary to what was given. Therefore the Duchess stole it (and hence the Duchess is lying, the Cheshire Cat is lying, and the Cook is telling the truth)."

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- Smullyan uses backtracking, which is the core of modern SAT solving!

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- Formulas in Conjunctive or Disjunctive Normal Form are easier to analyze.
- Converting a formula to CNF (DNF) can incur exponential running time.
- Tseytin transformation is efficient and provides an *equisatisfiable* formula in CNF.
- Modern SAT solvers begin by translating a formula to CNF using Tseytin transformation or similar techniques.