Propositional Logic Basics

COM 270: Introduction to Automated Deduction

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Propositional Logic

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- Suppose it is true that “If it is raining, then I have an umbrella”, and “I don’t have an umbrella”. Can we conclude that “It is not raining”? Propositional logic allows us to answer this question.

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- Suppose it is true that "If it is raining, then I have an umbrella", and "I don't have an umbrella". Can we conclude that "It is not raining"? Propositional logic allows us to answer this question.
- Solving questions like the one above can be hard (co-NP-complete).

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p	$\neg p$
T	F
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p	q	$p \vee q$
T	T	T
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p	q	$p \wedge q$
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p	q	$p \leftrightarrow q$
T	T	T
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F	T	F
F	F	T

Logical Equivalence (Implication Elimination Law)

Note that for any value of p and q, $p \to q$ always has the same value as $\neg p \vee q$:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\neg p$	$\neg p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

The same can be expressed as:

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F	T	T
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T	T	F	T
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$\neg p \vee q$	$p \to q$	$(p \to q) \leftrightarrow (\neg p \vee q)$
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Similarly, $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$.

Other Useful Logical Equivalences

$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$p \equiv \neg(\neg p)$	Double negation law
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws

$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

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- Suppose $(p \to q) \wedge \neg q$. Can we conclude $\neg p$?

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- Is this a good proof technique?

Problem: Truth Table Has Exponentially Many Rows

p	q	r	$\neg p$	$p \vee \neg p$	$\neg q$	$q \vee \neg q$	$\neg r$	$r \vee \neg r$	$(p \vee \neg p) \wedge (q \vee \neg q) \wedge (r \vee \neg r)$
T	T	T							
T	T	F							
T	F	T							
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T	T	T	F	T	F	T	F	T	T
T	T	F	F	T	F	T	T	T	T
T	F	T	F	T	T	T	F	T	T
T	F	F	F	T	T	T	T	T	T
F	T	T	T	T	F	T	F	T	T
F	T	F	T	T	F	T	T	T	T
F	F	T	T	T	T	T	F	T	T
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- Implication ($\rightarrow$) and biconditional ($\leftrightarrow$) are shorthands for what can already be expressed using the three basic operations above.
- One can prove logical equivalence by enumerating every possible variable assignment (i.e. by using a truth table).
- Enumeration is **guaranteed** to have exponential running time.